Introduction to Order Theory

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0 Introduction

Welcome, welcome here in the lecture hall at Jagiellonian University, I see X faces in front of me and welcome to those who see the lecture in the life streem or in the future from the records. This includes students in Kraków, Berlin and Warszawa.

For us, for Piotr Micek and myself this is a truely exciting enterprise. We look forward to present to you an introduction to combinatorial order theory with connections to graph theory, complexity theory, algebra and geometry.

A more detailed plan for the course can be found on the home pages. There you also find links to the literature and hopefully a growing collection of course notes.

After the lecture we will be open to questions reg. organization. Question regarding content are welcome anytime even in the chat

1. Fundamentals

1.1. Essential definitions and examples

The main character

$\boxed{\text{Partial order / poset / order}}$

$P = (X, \leq)$ X ground set

$\leq$ a relation on X obeying

- $x \leq x \quad \forall x \in X$ (reflexivity)
- $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity)
- $x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry)

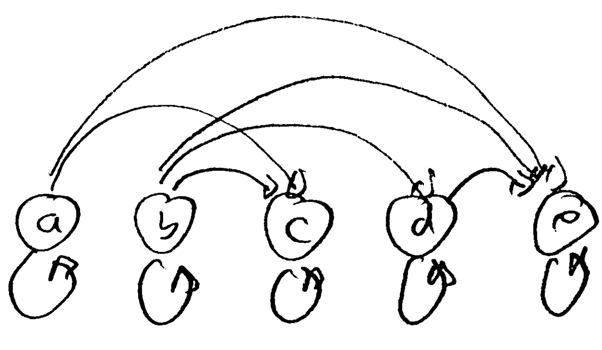
An example $X = \{a, b, c, d, e\}$
with relation $\{aa, ac, ae, bb, bc, bd, be$
$cc, dd, de, ee\}$

As matrix

	a	b	c	d	e
a	1	0	1	0	1
b		1	1	1	1
c			1	0	0
d	0			1	1
e					1

upper diagonal

As directed graph.



acyclic + loops

The $\boxed{\text{strict order}}$

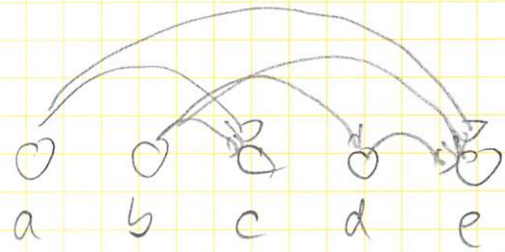
$P = (X, <)$ is strict if

- transitive
- asymmetric: not ($x<y$ and $y<x$)

matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 \\ & & 0 & 0 & 0 \\ & 0 & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

digraph acyclic

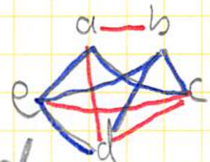


Pairs of elements

if $x<y$ or $x>y$ $\Rightarrow$ $\{x,y\}$ $\boxed{\text{comparable}}$ — notation $x \sim y$ (\sim)

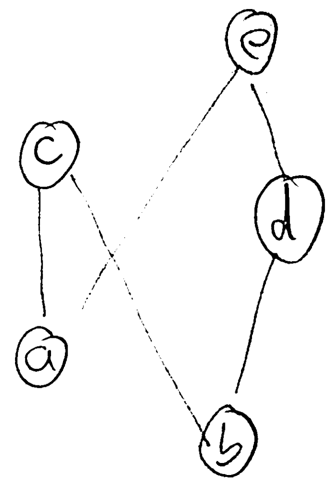
not($x<y$ or $x>y$) $\Rightarrow$ $\{x,y\}$ $\boxed{\text{incomparable}}$ — notation $x \parallel y$ (\parallel)

Taking the corresponding pairs as edges we obtain the $\boxed{\text{comparability graph}}$ and the incomparability graph, respectively (also $\boxed{\text{cocomparability graph}}$)



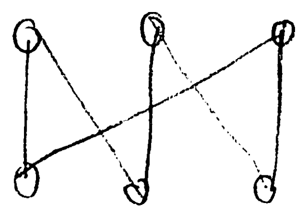
If $x<z$ and there is no y with $x<y<z$ then we say (x,z) is a cover of P — notation $x \prec z$ (\prec)

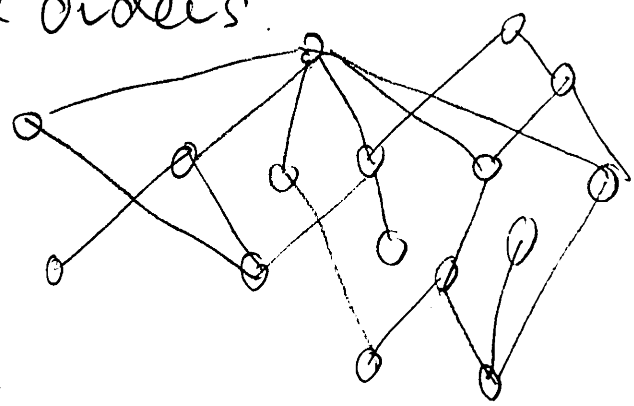
Based on $\boxed{\text{covers}}$ we can define the $\boxed{\text{diagram}}$ of P (the most useful type of drawing for orders.)



edges are cover pairs (x,z) drawn such that x is below y along the y-monotone edge

This enables us to quickly describe further partial orders.



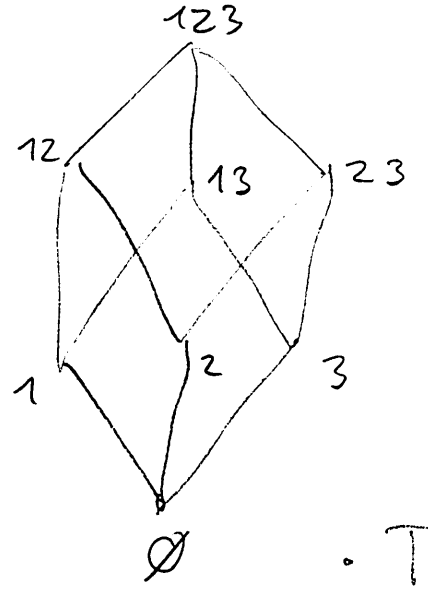


special orders

- Linear orders (total orders) or chains — similar to complete graph and clique
- Linear orders are important in computer science sorting "identifying an order"
- Sets of numbers $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, ordinal numbers

⚠ At this point we emphasize that unless explicitly stated all orders in this course are finite.

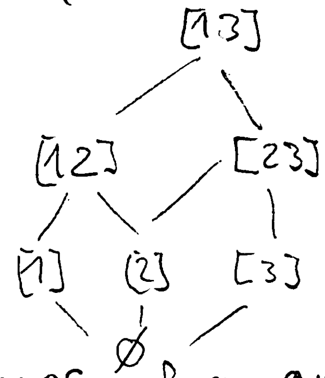
- Boolean Lattice



$\mathcal{B}_n = ([n], \subseteq)$

subset relation on all subsets of $[n] = \{1, 2 \dots n\}$

- The subset relation on any set family is an order relation



Examples: Intervals discs, balls, subtrees of a tree, subgroups of a group.

Proposition: Every finite order is a $\boxed{\text{containment order}}$ (comes from a subset relation)

proof: Let $P = (X, \leq)$

For an element a of P (ie $a \in X$) we define the $\boxed{\text{down set}}$ of a as $D[a] = \{x : x \leq a\}$

For $A \subseteq X$ the down set of A is $D[A] = \bigcup(D[a] ; a \in A) = \{x ; \exists a \in A \; x \leq a\}$

Let $Y = \{D[a] : a \in X\}$

Claim $P \cong (Y, \subseteq)$

isomorphic

Have to show $\exists$ bijection $X \leftrightarrow Y$ which respects the order relation

$D : x \mapsto D[x]$ is a bijection

injective: $x \neq y$ then $\underbrace{x < y \text{ or } x \| y}_{x \notin D[y]}$ or $\underbrace{x > y}_{y \notin D[x]}$

surjective by def.

order preserving: $x \leq y \Leftrightarrow D[x] \subseteq D[y]$

$\square$

The $\boxed{\text{downset lattice}}$ of a finite order

$$D(P) = \left(\{D[A] : A \subseteq X\}, \subseteq \right)$$

important object, distributive lattice

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

contains P as $\boxed{\text{subposet}}$ (induced)

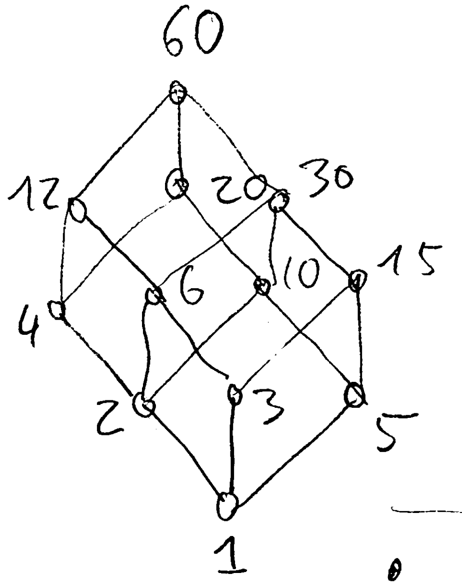
Def: $Q = (Y, \leq_Q)$ is a $\boxed{\text{subposet}}$ of $P = (X, \leq)$ iff $Y \subseteq X$ and $(\leq_Q) = (\leq_P) \cap (Y \times Y)$

Lattices: A finite poset is a $\boxed{\text{lattice}}$ if every subset has a $\boxed{\text{least upper bound}}$ ($\boxed{\text{join}}$ / supremum) and a $\boxed{\text{greatest lower bound}}$ ($\boxed{\text{meet}}$ / infimum

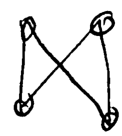
Example Lattice of divisors of $N \in \mathbb{N}$

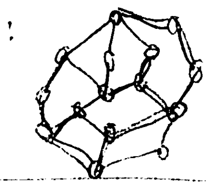
join = least common multiple

meet = greatest common divisor



Can recognize lattices by looking at diagram

- global Max and Min
  $\hat{1}$ $\hat{0}$
- no 

Ex: 

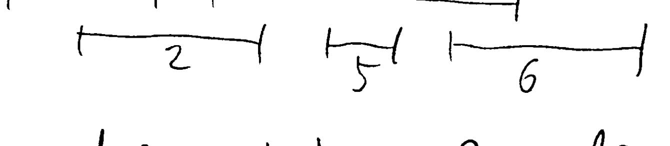
- Lattice of subgroups of a group
- Lattice of subspaces of a vector space

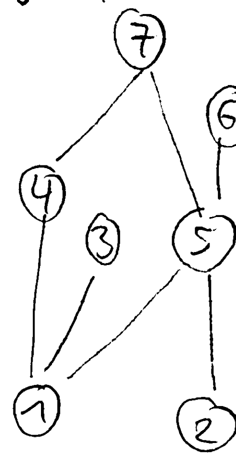
Interval Orders

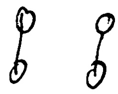
$\mathcal{I}$ family of intervals on $\mathbb{R}$

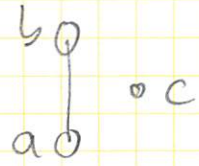
$I, J \in \mathcal{I}$ we define $I < J$

if $x < y$ $\forall x \in I$ $y \in J$

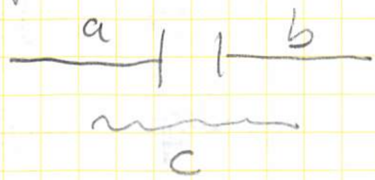
Ex 



Observation: Interval orders have no  as a subposet (this is called a 2+2)

Proof: 

representation



$\Rightarrow \forall x > c$ we have $x > a$

$\forall x < c$ we have $x < b$ $\square$

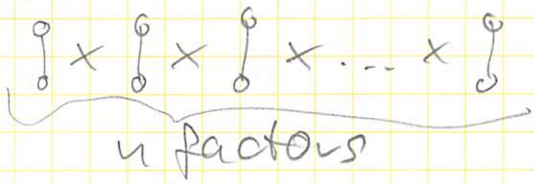
- Product orders

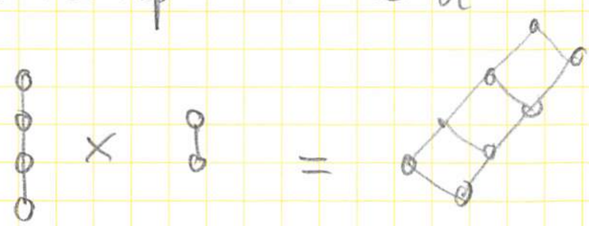
$P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ orders

$P \times Q$ is defined on $X \times Y$

with relations $(x, y) \leq (x', y')$

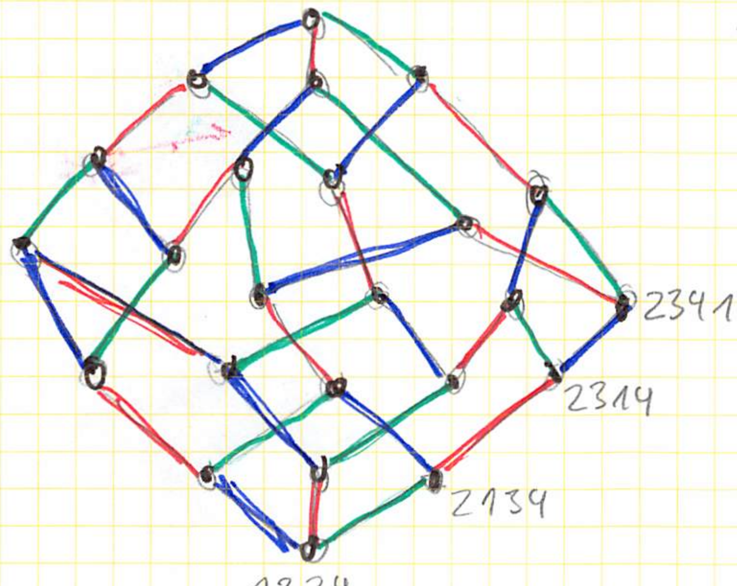
$\iff x \leq x'$ and $y \leq y'$

Examples $\mathbb{B}_n = 2^n =$ 



Combinatorial orders

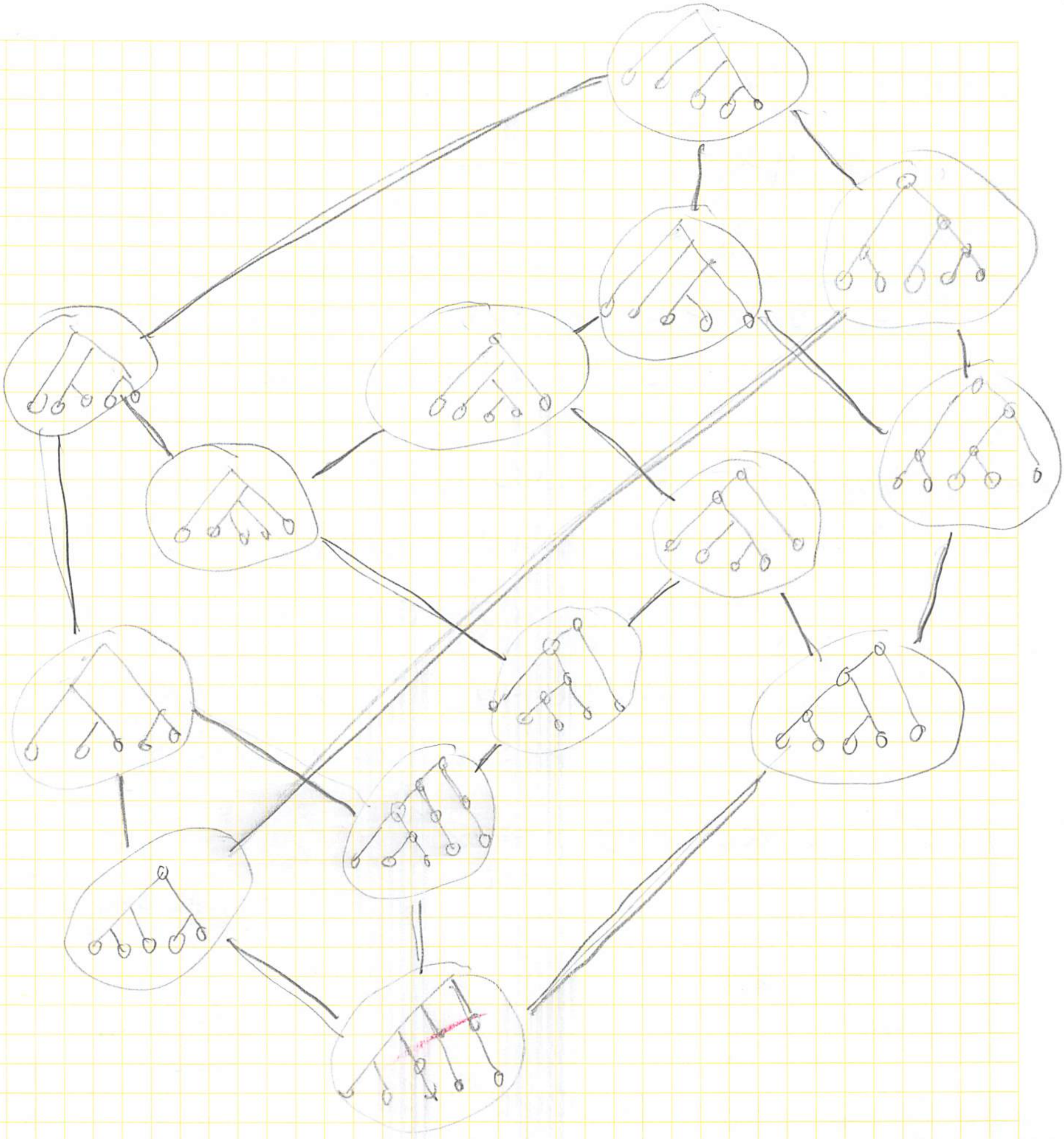
Permutations and inclusion of inversion sets

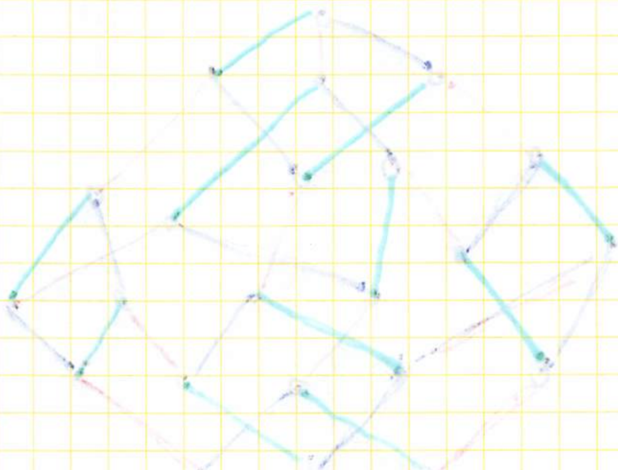


Permutahedron

Binary trees with rotation

Tamari lattice
associahedron





1.2 Chains and Antichains

$P=(X,\le)$ partial order

we already know what a chain of P is

- a chain is a subposet which is a linear order
- a <u>chain</u> is a subset C of X such that $x \sim y \quad \forall x y \in C$
- an antichain is a subset A of X such that $x \parallel y \quad \forall x y \in A$

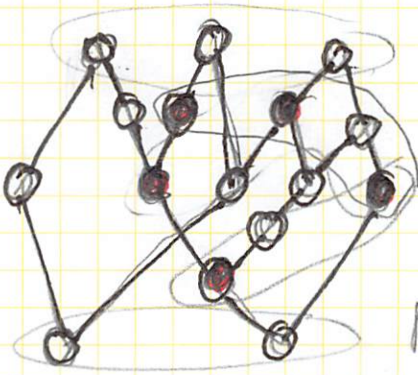
$\boxed{\text{Fact } A \text{ antich. } C \text{ chain} \Rightarrow |A \cap C| \le 1}$

An element x of P is maximal in P if there is no y with $x < y$

An element x of P is minimal if there is no y with $y < x$

If $Y \subseteq X$ then $\mathrm{Max}(Y)$ ~~resp $\mathrm{Min}(Y)$~~ denotes the set of maximal elements of the suborder P_Y induced by Y in P

Ex



Max(P)

Max(Y)

Min(Y)

Min(P)

Obs

$\mathrm{Max}(Y)$ and $\mathrm{Min}(Y)$ are antichains

Proposition: Antichains and down-sets of P are in bijection

Proof: A antichain $\Rightarrow$ $D(A)$ is a down-set

I a down-set (ideal) $\Rightarrow$ $Max(I)$ is an antichain.

$A = Max(D(A))$ $\qquad$ $I = D(Max(I))$

$\square$

A chain C of P is $\boxed{\text{maximal}}$ if there is no chain C' with $C \subset C'$ (proper subset)

Chain C is $\boxed{\text{maximum}}$ if there is no chain C' with $|C| < |C'|$

The $\boxed{\text{height}}$ of P is the size of a maximum chain

The $\boxed{\text{width}}$ of P is the size of a maximum antichain

Ex In the figure we have width $= 6$ height 5

Ex $\mathcal{B}_n$ every maximal chain has $n+1$ elements height$(\mathcal{B}_n) = n+1$

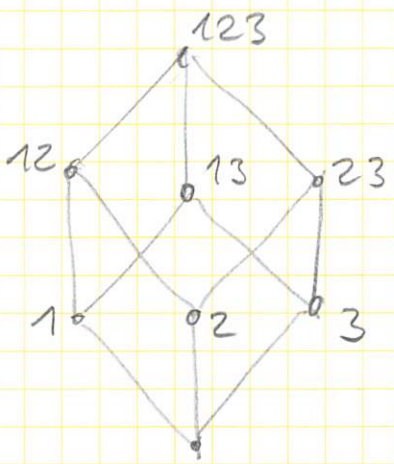
What about the width?

The levels of $\mathcal{B}_n$ yield antichains of sizes $\binom{n}{0}, \binom{n}{1}, \binom{n}{2} \dots \binom{n}{n}$

the sequence is unimodal the max is $\binom{n}{[\frac{n}{2}]}$ in the middle

Sperner's Theorem: $\mathrm{width}(\mathcal{B}_n) = \binom{n}{[\frac{n}{2}]}$

Proof



The maximal chains of $\mathcal{B}_n$ are in bijection to the permutations π of $[n]$

An element S of level k (a k-subset of $[n]$) is contained in exactly $k!(n-k)!$ maximal chains.

Let $\mathcal{A}$ be an antichain of $\mathcal{B}_n$

We double count pairs (A, C) with $A \in \mathcal{A}$, C a maximal chain and $A \in C$

$$\sum_{A \in \mathcal{A}} |A|!(n-|A|)! \leq n!$$

each chain contains at most one $A \in \mathcal{A}$

Let $\mathcal{A}$ contain p_k sets from level k

$$\Rightarrow \sum_k p_k \, k! \, (n-k)! \le n$$

$$\Rightarrow \boxed{\sum_k \frac{p_k}{\binom{n}{k}} \le 1}$$ LYM inequality

Since $\binom{n}{k} \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$ we get

$$\sum_k \frac{p_k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le \sum_k \frac{p_k}{\binom{n}{k}} \le 1$$

$$\Rightarrow |\mathcal{A}| = \sum_k p_k \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \qquad \square$$

The example of $\mathbb{B}_n$ suggests that for products $P \times Q$ we have

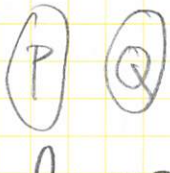
height$(P \times Q)$ = height(P) + height$(Q) - 1$

The width is harder to control.

Other compositions for partial orders

Parallel composition $P + Q$

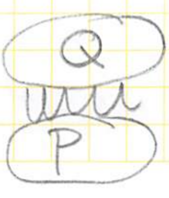
Stanley nennt das die direct sum



- disjoint ground sets $\to X_P \cup X_Q$
- union of relations

Serial composition $P \odot Q$

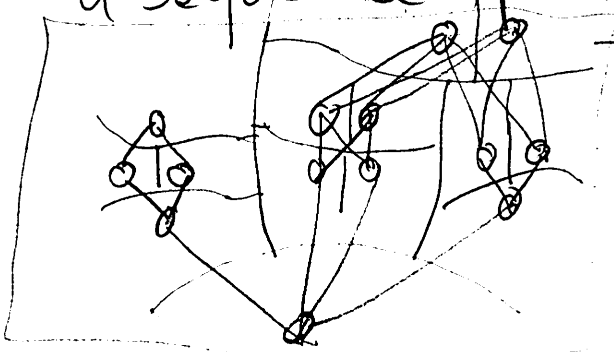
Stanley: ordinal sum



- disj. ground sets $\to X_P \cup X_Q$
- union of relations union all pairs (x, y) $x \in X_P$ $y \in X_Q$

$$h(P+Q) = \max(h(P), h(Q))$$
$$w(P+Q) = w(P) + w(Q)$$
$$h(P \odot Q) = h(P) + h(Q)$$
$$w(P \odot Q) = \max(w(P), w(Q))$$

Def: A poset is $\boxed{\text{series-parallel}}$ if it can be obtained from singletons via a sequence of $+$ and $\odot$ operations



A poset is a $\boxed{\text{weak order}}$ if it can be obtained from singletons via a sequence of $+$ op. followed by a sequence of $\odot$ op.

Dilworth's Theorem

Antichains are independent sets in the comparability graph. Comp(P)

Colorings of Com(P) are partitions of P into antichains.

Proposition: $P = (X, \leq)$ a partial order

$$\underbrace{\max(|C| : C \text{ chain in } P)}_{\text{height}(P)} = \min(|\mathcal{A}| \; \mathcal{A} \text{ antich. partition of } P)$$

Proof: Consider the following algorithm

```
algo APart(P)
  i ← 1
  while P ≠ ∅
    A_i ← Min P
    P ← P − A_i
    i ← i+1
```

Let $A_1 A_2 \dots A_k$ be the result

This is an antichain partition.

Claim: $\exists$ chain of size k in P

Pr: Choose $x_k \in A_k \Rightarrow \exists \underset{\in A_{k-1}}{x_{k-1}} < x_k$

$\Rightarrow$ . . . . . Result $x_1 < x_2 < \dots x_{k-1} < x_k$

the trivial ineq. $\leq$

take a maximum chain C each A_i can have at most 1 elem of C

□

Theorem [Dilworth] $P = (X, \leq)$ partial order

$$\underbrace{\max(|A| : A \text{ antichain})}_{\text{width}(P)} = \min(|\mathcal{C}| : \mathcal{C} \text{ chain part of } P)$$

Proof: Induction

Let A be a maximum antich.

$D(A)$ and $U(A)$ (up-set of A)

both have width $= |A| \Rightarrow \exists$ chain part.

glue them at A. This is a chain part of P

It may happen that $U(A)$ or $D(A)$ equals P for every maximum antichain A

$\Rightarrow$ can find $x \leq y$ in P such that

$P \setminus \{x, y\}$ has width $<$ width(P) induction again does the jos.

□

Note that the Proposition (dual of Dilworth's Thm)
as well as Dilw. Thm. imply that

$$|P| \leq h(P) \cdot w(P)$$

$A_1 .. A_h$ antichain part each $|A_i| \leq w$
$C_1 .. C_w$ chain part each $|C_j| \leq h$

Let $G = comp(P)$, we have

$h(P) = \omega(G)$ $\quad$ $w(P) = \alpha(G)$

min antichain part $= \chi(G)$ $\quad$ min chain part $= \Theta(G)$

A graph G is ω-perfect if $\forall H$ induced subgraph of G : $\omega(H) = \chi(H)$

The prop implies: G is ω-perfect
! induced subgraphs of a comp-graph are comparability graphs.

G is α-perfect if $\forall H$ induced subgr. of G
$\alpha(H) = \Theta(H)$ $\quad$ Dilworth implies G is α-perfect

Weak perfect graph theorem: Equivalent is

- G is ω-perfect
- G is α-perfect
- $\forall H$ ind. subgr $|V_H| \leq \alpha(H)\,\omega(H)$

Hence: the proposition implies Dilworth and vice-versa.

An dieser Stelle hätte Siehe
Erdös Szekeres gepasst Rückseite

Dilworth and related duality theorems

Dilworth – Max flow Min cut (FF)
– Bip. Matching (König-Egervary)
– Menger's theorem (all equivalent)

we show Bip Matching $\Rightarrow$ Dilworth

BMatching Thm: $G = (X, Y; E)$ bipartite

$$\max\left(|M| : \begin{array}{c} M \subseteq E \\ \text{matching} \\ \text{in } G \end{array}\right) = \min\left(|V| : \begin{array}{c} V \subseteq X \cup Y \\ \text{vertex cover} \\ \text{in } G \end{array}\right)$$

$P = (X, \leq)$ order build $B_P = (X', X'', E)$

For each $x \in X$ a copy $x' \in X'$ and a copy $x'' \in X''$

$(x', y'') \in E \iff x < y$ in P

A map: matchings of B_P $\rightarrow$ chain part of P

$$M \longrightarrow \mathcal{C}_M$$

$$|\mathcal{C}_M| = |X| - |M|$$

A : minimum vertex cover of B_P $\rightarrow$ antichain of P

$$V \longrightarrow \{x : V \cap \{x', x''\} = \emptyset\}$$

is an antichain

! minimum vertex cover never contains x' and x''

one of Pred(x)' and Succ(x)'' belongs to V

$\Rightarrow \quad |A_V| = |X| - |V|$

$|M| = |V| \Rightarrow |\mathcal{E}_M| = |A_V|$

Since bipartite matching is computationally easy ($< O(n^3)$) a minimum chain partition can also be found quickly.

Erdös Szekeres

$A = a_1 \dots a_{n^2+1}$ a seq of numbers
$\Rightarrow \exists$ increasing subseq of length $n+1$
or $\exists$ decreasing subseq of length $n+1$

Define a partial order P_A

$(i, a_i) \leq (j, a_j) \Leftrightarrow$ $i \leq j$ and $a_i \leq a_j$

chain: weakly increasing subseq of A
antichain: decreasing subseq of A

$|P_A| = n^2 + 1 \leq h(P_A)\,\omega(P_A)$

one of the two factors has to exceed n